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On the Forms of Sextic Scrolls of Genus Greater than One.

By Virgil Snyder.

The following paper is a direct continuation of my previous ones on the Sextic Scrolls which appeared in the Journal, Vol. XXV, pp. 58–96. The scroll is generated by the lines joining corresponding points of two directrix curves lying upon it.

§1.—*Sextic Scrolls of Genus* 2.

1. In this case the nodal curve is of order 8 and the simplest plane curve existing on the surface is a nodal quartic. Given two nodal quartics having the same characteristics which lie in different planes but have two points of intersection. These two curves can be put in (1, 1) point correspondence in such a way that the two points of intersection are self-corresponding elements. Lines joining corresponding points will generate a sextic scroll of genus 2. This is the general case, type I.

Consider the curve

$$z = 0, \quad y^2 = \frac{f_4(x)}{f_2(x)}.$$

Make any Cremona transformation upon it which will produce a quartic curve. The lines joining corresponding points can be rationally expressed in terms of hyper-elliptic functions.

In case the line of intersection of the two planes is a double generator, every plane passed through it will cut from the surface a nodal quartic having the same characteristics. The residual nodal curve of order 7 must therefore cut the double generator in six points. A plane containing the double generator and a single generator must accordingly contain either a double directrix and a single generator, or a triple directrix, hence the residual curve of order 7 must be composite. A triple generator cannot exist.

2. Consider a straight line and a nodal quartic. Put them in (2, 1) correspondence by making the points on the line projective with the pencil through the node. Connect these points with the points in which the line cuts the quartic again.

The equations of the line are

$$x = 0, \quad y = 0, \quad z = \mu,$$

and those of a variable generator connecting the point $(0, 0, \mu)$ with $(x_1, y_1, 0)$ are

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{\mu - z}{\mu},$$

wherein

$$y_1^2 = \frac{f_4(x_1)}{f_2(x_1)} \quad \text{and} \quad \mu = \frac{Ax_1 + B}{Cx_1 + D}.$$

If $\quad f_4(x) \equiv \alpha' x^4 + \beta' x^3 + \gamma' x^2 + \delta' x + \varepsilon', \quad f_2(x) \equiv \alpha x^2 + \beta x + \gamma,$

the equation may be written in the form

$$\begin{vmatrix} x^2\alpha' - y^2\alpha & x^2\beta' - y^2\beta & x^2\gamma' - y^2\gamma & \delta' x^2 & \varepsilon' x^2 & 0 \\ 0 & x^2\alpha' - y^2\alpha & x^2\beta' - y^2\beta & x^2\gamma' - y^2\gamma & \delta' x^2 & \varepsilon' x \\ Cz - Aw & Ax + Dz - Bw & Bx & 0 & 0 & 0 \\ 0 & Cz - Aw & Ax + Dz - Bw & Bxw & 0 & 0 \\ 0 & 0 & Cz - Aw & Ax + Dz - Bw & Bx & 0 \\ 0 & 0 & 0 & Cz - Aw & Ax + Dz - Bw & Bw \end{vmatrix} =$$

which contains the extraneous factor x.

The line x, y is a double directrix; the residual nodal curve of order 7 cuts the directrix once and every generator three times. Type II.

If the origin lies on the curve $\varepsilon' = 0$ and the double directrix is also a simple generator, the residual nodal curve is a quintic which cuts the multiple line and every other generator twice. Type III.

If, in the general case, $B = 0$, the equation may be written

$$\begin{aligned} &\varepsilon' x^2 (Cz - Aw)^4 - (Cz - Dw)^3 (Ax + Dz)\, \delta' x^2 \\ &\quad + (Cz - Aw)^2 (Ax + Dz)^2 (x^2\gamma - y^2\gamma') - (Cz - Aw)^3 (Ax + Dz)^3 (x^2\beta' - y^2\beta) \\ &\quad + (Ax + Dz)^4 (x^2\alpha' - y^2\alpha) = 0, \end{aligned}$$

which shows that the line $Ax + Dz = 0$, $Cz - Aw = 0$ is a four-fold directrix. The line $x = 0$, $z = 0$ is a double generator. This is the general (2, 4) correspondence with one double element between the points of two skew lines. Type IV.

When the directrix line passes through the node and also the vertex of the pencil, the two directrices become coincident. By a slight change of coordinates the equation may be written

$$xf_3(x, w)(xy - azw) + (xy - azw)^2 f_2(x, w) = x^2 f_4(x, w).$$

The line $x = 0$, $z = 0$ is a double generator as before, but the four-fold directrix is now $x = 0$, $w = 0$. Type V.

3. Let $f_4(x)$ be of the form $x(x - \varkappa)\phi_2(x)$, and let the pencil of lines through $x = \varkappa$, $y = 0$ be projective with the points of $x = 0$, $y = 0$ in such a way that the line of the pencil which passes through the origin corresponds to the origin. By using μ in the same sense as before, the equation of the pencil is

$$y = \frac{A\mu(x - \varkappa)}{C\mu + D},$$

from which the equation of the scroll generated by lines connecting the point $(0, 0, \mu)$ with the points in which the corresponding line of the pencil cuts the quartic again is

$$\begin{aligned} &y\,[Dy - A\varkappa z]\,f_2(Dxy - A\varkappa zx,\ Czy + Dyw - Axz) \\ &\qquad = x(Dx - \varkappa(Cz + Dw))\,\phi_2(Dxy - A\varkappa xz,\ Czy + Dyw - Axz). \end{aligned}$$

The line $x = 0$, $y = 0$ is a triple directrix; another triple directrix is $Dy - A\varkappa z = 0$, $Dx - \varkappa(Cz + Dw) = 0$. There are two double generators, $x = 0$, $Cz + Dw$ and $y = 0$, $z = 0$.

This is the general (3, 3) correspondence with two double elements between the points of two skew lines. Type VI.

In order to obtain the corresponding surface when the directrices are coincident, let (l, m) be any point on the curve. The pencil

$$(y - m) = \nu(x - l)$$

is to be made projective with the points of the line $x = l$, $y = m$ in such a way that the point $z = 0$ corresponds to the tangent

$$[f_2'(l) - f_4'(l)](x - l) + 2mf_2(l)(y - m) = 0.$$

The subsequent procedure is the same as before. Type VII.

If, in VI, x be put directly equal to zero, the equation reduces to type V. The reason is that the tangent to the curve at the origin passes through the node, thus imposing an extra condition.

A particular case of VII may be generated by letting m and l each become zero, but letting the pencil be defined by

$$x = \frac{A\mu y}{B\mu + C}.$$

The resulting equation is

$$y^2 f_2(Cx^2,\ Cxw + Bzx - Azy) = f_3(Cx^2,\ Cxw + Bzx - Azy),$$

in which $xf_3 = f_4$ when $\varepsilon' = 0$.

The line $z = 0$, $x = 0$ now counts for two consecutive double generators. Any plane section not containing the triple directrix nor this line will cut the latter in a tacnode of the curve of section.

Types IV to VII are the only sextic scrolls of genus 2 that are contained in a linear congruence.*

4. A surface containing a triple directrix but not contained in a linear congruence, may be generated as follows: Consider a pencil of conics passing through the node and three other basis points on the quartic. Each conic will cut the quartic in three further points. Make the pencil projective with the points of a line which cuts the quartic in one point—the correspondence being such that the point of intersection of the line and the quartic corresponds to the conic through the same point. Lines joining points of the fixed directrix line to the points in which the associated conic cuts the quartic will generate the scroll. The residual nodal curve is a quintic which cuts the directrix and every generator twice. Type VIII. The difference between types III and VIII lies in the configuration of the directrix. In III, two generators distinct from the directrix issue from each point of it; in VIII, there are three such generators and the directrix is not itself a generator.

If the quartic curve has a cusp, the line joining it to the corresponding point of the directrix will be a double generator. The residual is now a quartic cutting the directrix once and every generator twice. Type IX.

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* The most general quintic scroll having a tacnodal generator can be expressed in the form

$$y^3(yw - zx) = x(xy - a(yw - zx))^2.$$

It is unicursal.

In case the directrix in the general form (VIII) passes through the node, it becomes a triple line and simple generator. The residual nodal curve is a conic section cutting the directrix in one point. Type X.

Finally, let the line go through the node, but have the conic pass through four simple basis points on the quartic. As before, the node, regarded as a point on the directrix, is to correspond to the conic through it. The line is now a four-fold directrix. The residual is a conic which cuts it once. Type XI.

The configuration of nodal curves in X and XI suggests using these curves for directrices, the former by a (3, 2) correspondence with the point of intersection as a simple point, and the latter has the point of intersection for a double element.

Let
$$\mu^2 f_3(\lambda) + \mu\phi_3(\lambda) + \lambda f_2(\lambda) = 0,$$
wherein $\mu = \frac{yz}{yw - x^2}$, $\lambda = \frac{y}{x}$. The scroll is expressed by
$$yz^2 f_3(x, y) + z(yw - x^2)\,\phi_3(x, y) + (yw - x^2)^2 f_2(x, y) = 0.$$
Similarly, for the last type the correspondence is
$$\mu^2 f_4(\lambda) + \mu\lambda f_3(\lambda) + \lambda^2 f_2(\lambda) = 0,$$
from which
$$z^2 f_4(x, y) + z(yw - x^2) f_3(x, y) + (yw - x^2)^2 f_2(x, y) = 0.$$
The more particular case of a double directrix and a double generator is impossible for a surface of genus 2, since it would establish a (2, 2) correspondence between unicursal curves. The resulting scroll is given among those of genus 1 as type XII.

When the genus of a sextic scroll is greater than 1, the scroll cannot have a triple conic nor more thon one double conic as part of the nodal curve.

§2.—*Forms of Sextic Scrolls of Genus* 2.

I. c_8^2.
II. $d^2 + c_7^2$.
III. $(d^2, g^1) + c_5^2$.
IV. $d^4 + d^2 + g^2$.
V. $(d^4 \equiv d^2, g^2) + g^2$.
VI. $2d^3 + 2g^2$.
VII. $d^3 \equiv d'^3 + 2g^2$.
VIII. $d^3 + c_5^2$.
IX. $d^3 + c_4^2 + g^2$.
X. $(d^3, g^1) + c_2^2$.
XI. $d^4 + c_2^2$.

§3—*Sextic Scrolls of Genus* 3.

5. The nodal curve is now of order 7; the simplest curve lying on the surface is a non-singular quartic. Given two non-singular quartics having the same characteristics. Let them be in different planes but have two points of intersection. Establish a (1, 1) correspondence between the points of the two curves and connect corresponding points by straight lines. This scroll will be the general type of genus 3. The nodal curve is of order 7. Type I.

A double generator cannot exist without making the residual curve composite, neither can a double directrix for the latter would necessitate a (1, 2) correspondence between the points of a non-singular quartic and a straight line which intersects it, which is impossible. The nodal curve cannot consist of a double conic and a simple or composite nodal quintic, for if a plane be passed through the two generators which intersect at any point on the conic, the plane would cut the residual curve in six points, three on each generator.

The only remaining forms are accordingly those contained in a linear congruence.

6. Put a quartic curve and a line which intersects it in one point in (1, 3) correspondence by making the points of the line projective with a pencil whose vertex is any point on the quartic. Let the line of the pencil which cuts the directrix correspond to the point in which it cuts the directrix. The line will be a triple directrix. A double generator will exist in the plane of the quartic and the residual curve will be another triple directrix skew to the first and passing through the vertex of the pencil.

The general equation may be written in the form

$$y^3 f_3(z, w) + y^2 x\phi_3(z, w) + yx^2 z f_2(z, w) + x^3 z^2 f_1(z, w) = 0,$$

in which the triple directrices are $x = 0$, $y = 0$; $z = 0$, $w = 0$, and the double directrix is $y = 0$, $z = 0$. Type II.

When the vertex of the pencil lies on the directrix, the two directrices become coincident. The equation now is

$$y^2 f_4(x, y) + y f_3(x, y)(xz - wy) + f_2(x, y)(xz - wy)^2 + \varkappa(xz - wy)^3 = 0. \quad \text{Type III.}$$

7. The preceding are the only sextic scrolls of genus 2 on which quartic curves can lie. Two more forms exist which can be generated by replacing the

quartic by a quintic having a triple point at the vertex of the pencil. The directrix is now double and the residual is a four-fold directrix passing through the triple point. The equation is

$$y^2 f_4(z, w) + xy\phi_4(z, w) + x^2\psi_4(z, w) = 0. \qquad \text{Type IV.}$$

When the first directrix line also passes through the triple point, the two directrices become coincident. The equation is

$$f_6(x, y) + f_4(x, y)(xz - wy) + f_2(x, y)(xz - yw)^2 = 0. \qquad \text{Type V.}$$

§4.—*Nodal Curves of Sextic Scrolls of Genus* 3.

I. c_7^2. II. $2d^3 + g^2$. IV. $d^4 + d^2$.
III. $d^3 \equiv d'^3 + g^2$. V. $d^4 \equiv (d^2, g^2)$.

§5.—*Sextic Scrolls of Genus* 4.

8. Since any plane section must be of genus 4, it follows that any plane section containing two generators can only contain generators or directrix lines. Since the most general (2, 4) correspondence between the points of two lines generates a scroll of genus 3, it follows that the (3, 3) correspondence is the only one possible.

Given a quintic curve with two double points. Let a directrix line passing through one of them be made projective with a pencil lying in the plane of the quintic and vertex at the other node, the line joining the two nodes corresponding to the node on the directrix.

The equation of the resulting scroll is

$$y^3 f_3(z, w) + y^2 x\phi_3(z, w) + yx^2\psi_3(z, w) + x^3\chi_3(z, w) = 0. \qquad \text{Type I.}$$

When the two nodes become consecutive, forming a tacnode, the two directrices become coincident. The equation now is

$$f_4(x, y) + f_4(x, y)(zx - yw) + f_2(x, y)(zx - yw)^2 + \varkappa(zx - yw)^3 = 0. \qquad \text{Type II.}$$

9. These are the only forms of sextic scrolls of genus 4:

I. $2d^3$. II. $d^3 \equiv d'^3$.

10. In general, the highest genus that a scroll of order $2n$ can have is $(n-1)^2$. The nodal curve consists of two skew or coincident n-fold directrix lines. The simplest plane curve that can exist on the surface is one of order $2n-1$ having two $(n-1)$-fold points. A scroll of order $2n$ and having $(n-1)^2$ double generators exists, and is unicursal.

Similarly, if the order of the scroll be $2n-1$, the highest genus is $n(n-1)$. The orders of the two directrices are $n, n-1$. The order of the simplest plane curve is $2n-2$ and it has an $(n-1)$-fold and another $(n-2)$-fold point. A unicursal scroll of order $2n-1$ may have $n(n-1)$ double generators.

CORNELL UNIVERSITY, *August*, 1902.